

Erratum: Linear projections and successive minima

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Let E be a finitely generated projective modules over the ring of integers in a number field K , h an hermitian metric on $E \otimes_{\mathbb{Z}} \mathbb{C}$, and $X_K \subset \mathbb{P}(E_K^\vee)$ a smooth projective curve. In [2], Prop. 1, we formulated an inequality relating the successive minima of the euclidean lattice (E, h) to the Faltings height of X_K .

As explained in §1 below, the proof of this proposition is incorrect. We shall prove another result instead (see §2.1, Theorem 1).

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1 Erratum

The proof of Proposition 1, hence Theorem 2, in [2] is incorrect. Indeed, §2.5 and §2.7 in *op.cit* contain a vicious circle: the definition of the filtration V_i , $1 \leq i \leq n$, in §2.5 depends on the choice of the integers n_i , when the definition of the integers n_i in §2.7 depends on the choice of the filtration (V_i) . Thus, only Theorem 1 and Corollary 1 in [2] are proved.

2 An inequality

2.1

Let K be a number field, O_K its ring of algebraic integers and $S = \text{Spec}(O_K)$ the associated scheme. Consider an hermitian vector bundle (E, h) over S . Define the i -th successive minima μ_i of (E, h) as in [2] §2.1. Let $X_K \subset \mathbb{P}(E_K^\vee)$ be a smooth, geometrically irreducible curve of genus g and degree d . We assume that $X_K \subset \mathbb{P}(E_K^\vee)$ is defined by a complete linear series on X_K , and that $d \geq 2g + 1$. The rank of E is thus $N = d + 1 - g$. Let $h(X_K)$ be the Faltings height of X_K ([2] §2.2).

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For any positive integer $i \leq N$ we define the integer f_i by the formulas

$$\begin{aligned} f_i &= i - 1 & \text{if } i - 1 \leq d - 2g, \\ f_i &= i - 1 + \alpha & \text{if } i - 1 = d - 2g + \alpha, \quad 0 \leq \alpha \leq g, \end{aligned}$$

and $f_N = d$.

Fix a natural integer $t \leq N - 3$. When $i \geq t + 2$ we let

$$A_i(t) = \frac{f_i^2}{(i - t - 1) f_i - \sum_{j=t+2}^{i-1} f_j},$$

and

$$A(t) = \max_{t+2 \leq i \leq N} A_i(t).$$

Theorem 1. *Let t be an integer such that $0 \leq t \leq N - 3$. There exists a constant $c(N)$ such that the following inequality holds:*

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + (2d - A(t)(N - t - 1)) \mu_1 + A(t) \sum_{\alpha=1}^{N-t-1} \mu_\alpha + c(N) \geq 0.$$

2.2

To prove Theorem 1 we start by the following variant of theorem 4.1 in [1].

Proposition 1. *Fix an increasing sequence of integers $e_1 \leq e_2 \leq \dots \leq e_N$ such that $e_i = 0$ if $i \leq t + 1$. Set*

$$T(r_1, \dots, r_N) = \min_{0=i_1 < \dots < i_\ell = N} \left[\sum_{j=1}^{\ell-1} (r_{i_j} - r_{i_{j+1}}) \frac{e_{i_j} + e_{i_{j+1}}}{2} \right].$$

Subject to the constraints $\sum_{j=t+1}^N r_j = 1$, $r_1 = r_2 = \dots = r_{t+1}$, and $r_1 \geq r_2 \geq \dots \geq r_N = 0$, the maximal value of $T(r_1, \dots, r_N)$ is

$$T_{\max} = \frac{1}{2} \max_{t+2 \leq i \leq N} \frac{e_i^2}{(i - t - 1) e_i - \sum_{j=t+2}^{i-1} e_j}.$$

Proof. As in [1], *loc.cit.*, we may first assume that $T = 1$ and seek to minimize the sum of the r_j 's. If we graph the points (e_j, r_j) , T is the area of the Newton polygon they determine in the first quadrant. Moving the points not lying on the polygon down onto it only reduces $\sum_{j=t+1}^N r_j$, so we may assume that all the points

actually lie on the polygon. In particular we may assume that $(e_j, r_j) = (0, r_j)$ lies on this polygon when $1 \leq j \leq t+1$. For such r_i we have

$$T(r_1, \dots, r_N) = \sum_{i=1}^{N-1} (r_i - r_{i+1}) \frac{e_i + e_{t+1}}{2}.$$

Let $\sigma_i = r_{i-1} - r_i$, $i = 2, \dots, N$. The condition that the points (e_i, r_i) lie on their Newton polygon and that the r_i decrease becomes, in terms of the σ_i ,

$$\frac{\sigma_{t+2}}{e_{t+2} - e_t} \geq \frac{\sigma_{t+3}}{e_{t+3} - e_{t+2}} \geq \dots \geq 0. \quad (1)$$

Furthermore

$$\sigma_2 = \dots = \sigma_{t+1} = 0.$$

The constraint $\sum_{j=t+1}^N r_j = 1$ becomes

$$\sum_{j=t+2}^N (j - t - 1) \sigma_j = 1. \quad (2)$$

In the $(N - t - 2)$ -dimensional subspace of the points $\sigma = (\sigma_{t+2}, \dots, \sigma_N)$ defined by (2), the inequality (1) defines an $(N - t - 2)$ simplex. The linear function

$$T = \sum_{j=t+2}^N \sigma_j \frac{e_{j-1} + e_j}{2}$$

must achieve its maximum on this simplex at one of the vertices, *i.e.* a point where, for some i and α , we have

$$\alpha = \frac{\sigma_{t+2}}{e_{t+2} - e_{t+1}} = \dots = \frac{\sigma_i}{e_i - e_{i-1}} > \frac{\sigma_{i+1}}{e_{i+1} - e_i} = \dots = 0.$$

We get

$$\sigma_j = \begin{cases} \alpha(e_j - e_{j-1}) & \text{if } t+2 \leq j \leq i \\ 0 & \text{else.} \end{cases}$$

Then

$$\begin{aligned} 1 &= \sum_{j=t+2}^N (j - t - 1) \sigma_j = \alpha \sum_{j=t+2}^i (j - t - 1)(e_j - e_{j-1}) \\ &= \alpha \left((i - t - 1) e_i - \sum_{j=t+2}^{i-1} e_j \right), \end{aligned}$$

hence

$$\alpha = \left((i - t - 1) e_i - \sum_{j=t+2}^{i-1} e_j \right)^{-1},$$

and, as in [1] *loc.cit.*, we deduce that

$$T(r_1, \dots, r_N) = \frac{1}{2} \frac{e_i^2}{(i-t-1)e_i - \sum_{j=t+2}^{i-1} e_j}.$$

Corollary 1. *Let $t \geq 0$ be an integer less than $N - 2$. Fix an increasing sequence of integers $e_1 \leq e_2 \leq \dots \leq e_N$ and a decreasing sequence of numbers $r_1 \geq r_2 \geq \dots \geq r_N$. Assume that $e_1 = e_2 = \dots = e_{t+1} = 0$ and $r_1 = r_2 = \dots = r_{t+1}$. Let*

$$S = \min_{0=i_0 < \dots < i_\ell = N} \sum_{j=0}^{\ell-1} (r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}}).$$

Then

$$S \leq \left[\max_{t+2 \leq i \leq N} \left(\frac{e_i^2}{(i-t-1)e_i - \sum_{j=t+2}^{i-1} e_j} \right) \right] \sum_{j=t+2}^N (r_j - r_N).$$

2.3

We come back to the situation of Theorem 1. For every complex embedding $\sigma : K \rightarrow \mathbb{C}$, the metric h defines a scalar product h_σ on $E \otimes_{O_K} \mathbb{C}$. If $v \in E$ we let

$$\|v\| = \max_{\sigma} \sqrt{h_\sigma(v, v)}.$$

Choose N elements x_1, \dots, x_N in E , linearly independent over K and such that

$$\log \|x_i\| = \mu_{N-i+1}, \quad 1 \leq i \leq N.$$

Let $y_1, \dots, y_N \in E_K^\vee$ be the dual basis of x_1, \dots, x_N . Let $A(d)$ be the constant appearing in [2], Theorem 1.

Lemma 1. *Assume $0 \leq t \leq N - 3$. We may choose integers n_i , $i = 1, \dots, t+1$, such that*

- i) *For all $i \leq t+1$, $|n_i| \leq A(d) + d$*
- ii) *Let $w_i = y_i + n_i y_{i+1}$ if $i \leq t+1$ and $w_i = y_i$ if $i \geq t+2$.*

Let $\langle w_1, \dots, w_i \rangle \subset E_K^\vee$ be the subspace spanned by w_1, \dots, w_i , and

$$W_i = E_K^\vee / \langle w_1, \dots, w_i \rangle$$

($W_0 = E_K^\vee$). Then, when $i \leq t+1$, the linear projection from $\mathbb{P}(W_{i-1})$ to $\mathbb{P}(W_i)$ does not change the degree of the image of X_K .

Proof. The image of w_i in $\mathbb{P}(W_{i-1})$ is the center of the linear projection $\mathbb{P}(W_{i-1}) \dots \rightarrow \mathbb{P}(W_i)$. When $n \neq m$ are two integers, the images of the vectors $y_i + n y_{i+1}$ and $y_i + m y_{i+1}$ in W_{i-1} are K -linearly independent, hence their images in $\mathbb{P}(W_{i-1})$ are distinct. Lemma 1 follows from this and [2], Theorem 1 and Corollary 1.

2.4

Let $(v_i) \in E_K^N$ be the dual basis of (w_i) . Since the matrix expressing (w_i) in terms of (y_i) has integral coefficients and determinant one, the same is true for the matrix expressing (v_i) in terms of (x_i) . Furthermore these matrices have coefficients bounded by a function of N . This implies that there exists a constant $c_1(N)$ such that

$$\log \|v_i\| \leq \mu_N + c_1(N)$$

when $1 \leq i \leq t+1$.

On the other hand, when $t+2 \leq i \leq N$, we get $v_i = x_i$, hence

$$\log \|x_i\| = \mu_{N+1-i} \quad \text{if } i \geq t+2.$$

2.5

Let e_i be the drop in degree of X_K when projected to W_i . From Lemma 1 ii) we get

$$e_1 = e_2 = \dots = e_{t+1} = 0.$$

Let

$$r_i = \begin{cases} \mu_N + c_1(N) & \text{if } i \leq t+1 \\ \mu_{N+1-i} + c_1(N) & \text{if } i \geq t+2. \end{cases} \quad (3)$$

From Corollary 1 we deduce, as in [2] pp. 52–53, that there exists a constant $c_2(N)$ such that

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + 2d r_N \geq -A(t) \sum_{j=t+2}^N (r_j - r_N) - c_2(N),$$

where

$$A(t) = \max_{t+2 \leq i \leq N} \frac{f_i^2}{(i-t-1)f_i - \sum_{j=t+2}^{i-1} f_j}.$$

From (3) we deduce that there exists $c(N)$ such that

$$\frac{h(X_K)}{[K : \mathbb{Q}]} + (2d - A(t)(N-t-1))\mu_1 + A(t) \sum_{\alpha=1}^{N-t-1} \mu_\alpha + c(N) \geq 0.$$

Theorem 1 is proved.

References

- [1] Morrison, I. Projective stability of ruled surfaces. *Invent. Math.* 56, 269-304 (1980).
- [2] Soulé, C. Linear projections and successive minima. *Nagoya Math. J.* 197 (2010), 45-57.